Numerical relativity and spectral methods

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Numerical relativity

In the strong field regime gravity is described by Einstein equations. It is a set of 10 highly coupled non-linear equations.

Two families of methods

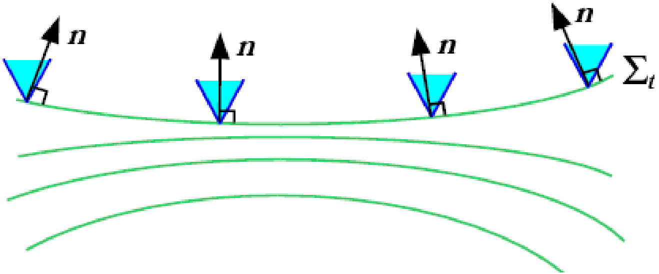
- Semi-analytical : developments in terms of v/c (pN expansions).
- Use of computers : numerical relativity

Fields of application

- Coalescence of compact binaries.
- Supernovae explosions.
- Structure of compact objects (magnetized neutrons stars, bosons stars...)
- Critical phenomena.
- Stability of ADS spacetimes (geons)
- and more...

3+1 formalism

Write Einstein equations in a way that is manageable by computers.
It is a way of explicitly splitting time and space.
Spacetime is foliated by a family of spatial hypersurfaces Σ_t.



- Coordinate system of Σ_t : (x_1, x_2, x_3).
- Coordinate system of spacetime : (t, x_1, x_2, x_3).

Metric quantities

The line element reads

$$\mathrm{d}s^2 = -\left(N^2 - N^i N_i\right)\mathrm{d}t^2 + 2N_i \mathrm{d}t\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

Various functions

- Lapse N, shift $\vec{N}$ and spatial metric γ_{ij}.
- They are all temporal sequences of spatial quantities.
- Lapse and shift are coordinate choice.

Projection of Einstein equations

Type	Einstein	Maxwell
Constraints	Hamiltonian $R + K^2 - K_{ij}K^{ij} = 0$ Momentum : $D_j K^{ij} - D^i K = 0$	$\nabla \cdot \vec{E} = 0$ $\nabla \cdot \vec{B} = 0$
Evolution	$\frac{\partial \gamma_{ij}}{\partial t} - \mathcal{L}_{\vec{N}} \gamma_{ij} = -2NK_{ij}$ $\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\vec{N}} K_{ij} = -D_i D_j N +$ $N\left(R_{ij} - 2K_{ik}K^k_j + KK_{ij}\right)$	$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon_0 \mu_0}\left(\vec{\nabla} \times \vec{B}\right)$ $\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$

R_{ij} is the Ricci tensor of γ_{ij} and D_i the covariant derivative associated with γ_{ij}.
K_{ij} is called the extrinsic curvature.

A two steps problem

Evolution problem

- Given initial value of $\gamma_{ij}\,(t=0)$ and $K_{ij}\,(t=0))$ use the evolution equations to determine the fields at later times.
- Similar to writing Newton's equation as $\partial_t x = v; \partial_t v = f/m$.
- Must ensure stability and accuracy.
- Must choose the lapse and shift in a clever way.

Initial data

- $\gamma_{ij}\,(t=0)$ and $K_{ij}\,(t=0)$ are not arbitrary but subject to the constraint equations.
- Is is a set of four elliptic coupled equations.
- Needs to make the link between a given physical situation and the mathematical objects γ_{ij} and K_{ij}

Both steps are equally important and complicated.

Spectral expansion

Given a set of orthogonal functions Φ_i on an interval Λ, spectral theory gives a recipe to approximate f by

$$f \approx I_N f = \sum_{i=0}^{N} a_i \Phi_i$$

Properties

- the Φ_i are called the basis functions.
- the a_i are the coefficients.
- Multi-dimensional generalization is done by direct product of basis.

Usual basis

- Orthogonal polynomials : Legendre or Chebyshev.
- Trigonometrical polynomials (discrete Fourier transform).

Coefficient and configuration spaces

There exist $N+1$ point x_i in Λ such that

$$f(x_i) = I_N f(x_i)$$

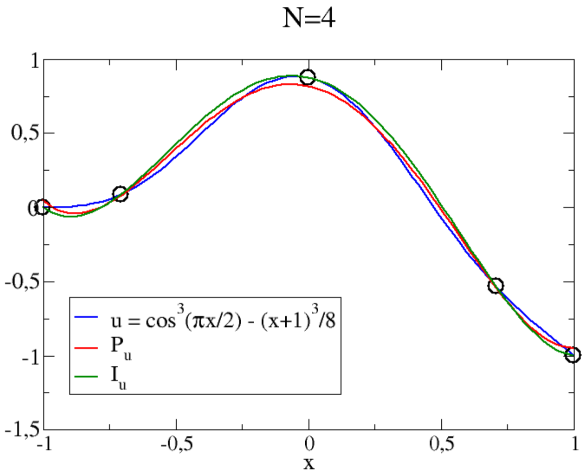
Two equivalent descriptions

- Formulas relate the coefficients a_i and the values $f(x_i)$
- Complete duality between the two descriptions.
- One works in the coefficient space when the a_i are used (for instance for the computation of f').
- One works in the configuration space when the $f(x_i)$ are employed (for the computation of $\exp(f)$)

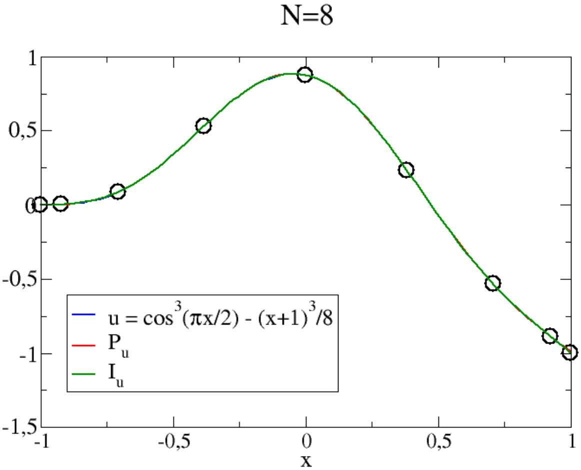
Spectral convergence

- If f is $\mathcal{C}^\infty$, then $I_N f$ converges to f faster than any power of N.
- Much faster than finite difference schemes.
- For functions less regular (i.e. not $\mathcal{C}^\infty$) the error decreases as a power-law.
- Spectral convergence can be recovered using a multi-domain setting.

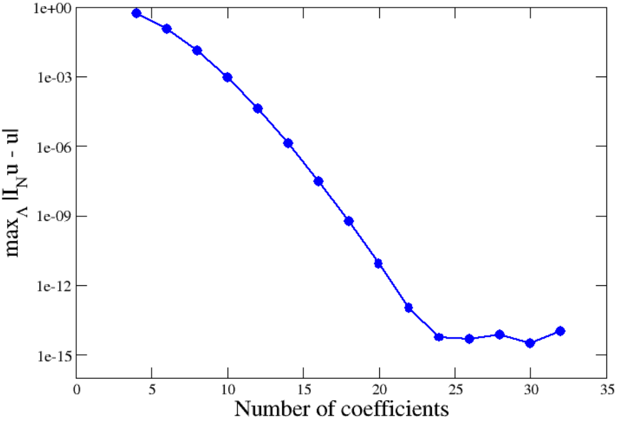
Collocation points



Collocation points



Spectral convergence



Weighted residual methods

Let $R = 0$ be a field equation (like $\Delta f - S = 0$). The weighted residual method provides a discretization of it by demanding that

$$(R, \xi_i) = 0 \quad \forall i \leq N$$

Properties

- $(,)$ denotes the same scalar product as the one used for the spectral expansion.
- the ξ_i are called the test functions.
- For the $\tau-$method the ξ_i are the basis functions (i.e. one works in the coefficient space).
- Some of the last residual equations must be relaxed and replaced by appropriate matching and boundary conditions to get an invertible system.
- Additional regularity conditions can be enforced by a Galerkin method.

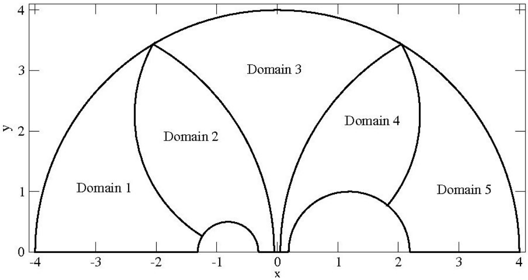
What is KADATH ?

KADATH is a library that implements spectral methods in the context of theoretical physics.

- It is written in C++, making extensive use of object oriented programming.
- Versions are maintained via Subversion.
- Minimal website : *http ://luth.obspm.fr/~luthier/grandclement/kadath.html*
- The library is described in the paper : *JCP* **220**, *3334 (2010)*.
- Designed to be very modular in terms of geometry and type of equations.
- LateX-like user-interface.
- More general than its predecessor LORENE.

Basic features

- Multi-domain approach (spherical, bispherical, cylindrical, periodic in times ...)
- In principle applicable to any kind of equations.
- The unknowns are the coefficients of all the fields in all space $\vec{u}$.
- The equations are dealt with using the weighted residuals method.
- The solution of the resulting discrete system $\vec{F}(\vec{u}) = 0$ is sought by a Newton-Raphson method.



The Newton-Raphson iteration

Properties

- Start from an initial guess and converges to the solution iteratively.
- Is the multi-dimensional generalization of Newton method.
- At each step : inversion of the Jacobian linear system $Jx = S$.

Implementation in KADATH

- The Jacobian is computed numerically by means of an automatic differentiation technique.
- It is obtained column by column (easy to parallelize).
- The inversion is also parallel and done via `SCALAPACK`
- The Jacobian can be very big ($200,000 \times 200,000$), especially for 3D problems.
- Can require several thousands of processors (used on various supercomputers like Curie).

Boson star model

A boson star is described by a complex scalar field ϕ coupled to gravity.
Is an alternative to black holes, especially in the context of supermassive objects at the center of galaxies.
The field is invariant under a $U\left(1\right)$ symmetry :

$$\phi \longrightarrow \phi \exp\left(i\alpha\right).$$

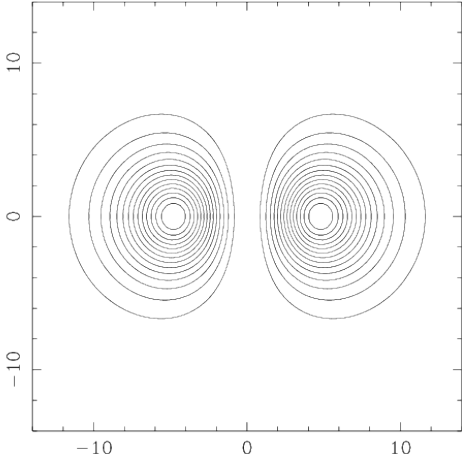
The Lagrangian of the matter is given by

$$\mathcal{L}_M = -\frac{1}{2}\left[g^{\mu\nu}\nabla_\mu\bar{\phi}\nabla_\nu\phi + V\left(|\phi|^2\right)\right].$$

Structure of the solution

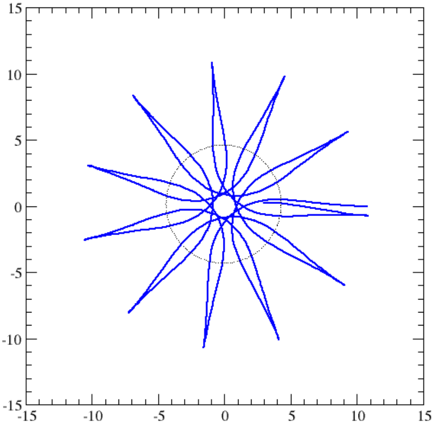
- One seeks solutions such that $\phi = \phi_0 \exp\left[i\left(\omega t - k\varphi\right)\right]$.
- ϕ_0 and the metric fields depend only on (r, θ)
- The solutions are found using the *Polar* space of `KADATH` , for axisymmetric configurations.
- One solves Einstein equations coupled to the Klein-Gordon one.
- Each BS is labelled by k and ω.

Field configuration

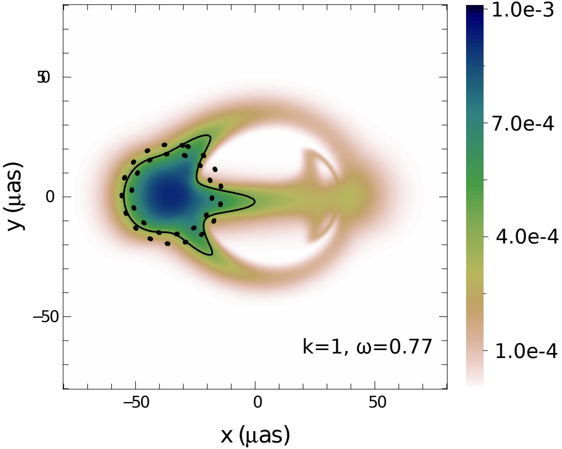


Case $k = 2$, $\omega = 0.8$

Peculiar types of orbits



Accretion disk around a boson star



Polish doughnut around a $k = 1$, $\omega = 0.77$ BS

Quasi-circular binaries

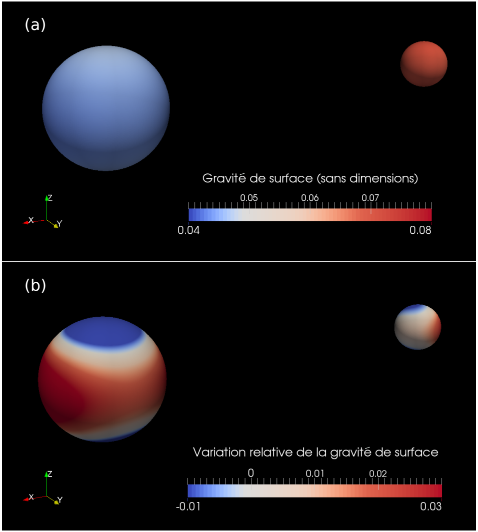
Properties

- Assume the orbits are closed and circular.
- Not exact due to gravitational waves emission.
- Enables to remove time by $\partial_t \longrightarrow \Omega \partial_\varphi$.
- Good approximation for widely separated objects.
- GW can be killed by the so-called conformal flatness approximation $\gamma_{ij} = \Psi^4 f_{ij}$.

Mathematical problem

- 5 unknown fields.
- 5 coupled, non-linear, elliptic equations.
- non-trivial boundary conditions on the horizons.
- Solved using the bispherical coordinates of `KADATH` .

Application : zeroth law of BBH thermodynamics



Time evolutions

Current status in KADATH

- No explicit time evolution.
- Always some symmetry in time (stationnarity, periodicity).

Spectral methods in time

- Not used by most of the groups (Runge-Kutta in time).
- Possible to implement :
  - Use Chebyshev polynomials in time
  - Impose the value of $f\,(t=0)$ and $\partial_t f\,(t=0)$.
  - Integrate on $[0, \Delta T]$.
  - Repeat for latter times
- Conditions on ΔT ?
- Long compared to Runge-Kutta. Useful ?

Iterative solvers

Most expensive part : computation and inversion of the Jacobian.

Alternative method for solving $Jx = S$

- Krylov subspace : assume $x \approx \sum_n J^n S$.
- Finds x iteratively.
- Each iteration requires to be able to compute products like $J \times f$.
- Various incarnations (Bicgstab, GMRES).

Properties

- Faster if the number of iterations $\ll$ size of the Jacobian.
- But convergence is far from being guarantied.
- Need for a preconditionner M (solve $MJx = MS$).
- Requires some fine-tuning of the algorithm

Probably not general enough for `KADATH` but worth investigating in some given cases.

Conclusions

- After years of struggle numerical relativity is able to produce meaningful results.
- Still some work (initial data, various fields configurations, realistic simulations).
- Spectral methods are a powerful tool.
- `KADATH` enables their use in a very modular manner.
- Several applications but still ongoing work.